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EQUILIBRIUM EQUATIONS AND A COMPUTATION METHOD FOR MATRIX DIFFERENTIAL GAMES (MDG)

by

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and

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It is shown that Danskin's omega-value of a matrix-differential game can be obtained as a solution of a finite system of equations, each involving a single maximization of minimization. A computation method is proposed based on these equations.

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1. Introduction

In a Matrix-Differential Game, (MDG) a finite payoff matrix A is given, and player I (II) tries to maximize (minimize) the payoff by picking the proper row (column). However, the choices of the two players are not made simultaneously, so that the conventional result that the game value is obtained when the two players use optimal mixed strategies does not hold. Instead, a number σ, with $0 \leq \sigma \leq 1$, is given, with the interpretation that σ is the relative reaction time of player I. The elements of A are interpreted as payoff rates, with the payoff being at the rate $a(i,j)$ for a time σ if player II has just picked column j (after which player I picks a new row), or for a time $1-\sigma$ if player I has just picked row i (after which player II picks a new column). Play is assumed to go on indefinitely, and the payoff $\Omega(\sigma)$ is defined to be the long run average of the payoff rate. The words "long run" are of course relative to the length of the time unit.

The idea of an MDG was invented by Danskin [1], who observed that the ability to solve MDG's is a prerequisite for being able to solve for the Ω-value of a differential game, where the payoff matrix of the MDG would be a Hamiltonian function involving the equations of motion. In fact, it is non-separable differential games that provide most of the interest in MDG's, since it can be shown that optimal strategies for playing MDG's do not involve choosing random numbers. The spectre of the two players trying to choose independent random numbers continuously in a differential game has given rise to the Ω-formulation (along with several other [2], [3], and [4]), where the need for mixed strategies is eliminated by having the players "take turns" in some sense or other.

Formally, an $N \times M$ MDG is a stochastic game with perfect information, $2MN$ positions (the position depends on whose move it is, hence the factor of 2), and non-zero stop probabilities. The existence of the Ω-value and of stationary optimal strategies has therefore been proved by Gillete in his paper onthe subject [5]. (His proof contained an error that was subsequently corrected by Liggett and Lippman [6]). Our Theorem 1 goes on to state that the Ω-value and the optimal stationary strategies must satisfy certain "equilibrium equations" (2.6 and 2.7) that form the basis of the computational procedure discussed in chapter 3. Theorem 2 shows that our Ω-value is the same as Danskin's, and Theorem 3 establishes some elementary properties for the Ω-value of a MDG. These three theorems, together with the computational procedure of chapter 3, are the content of this technical report.

2. The Equilibrium Equations

Throughout this report the symbol $A = [a(i,j)]$ will denote an $N \times M$ matrix with real entries $a(i,j)$; $i = 1,\ldots,N$; $j = 1,\ldots,M$. We further reserve the letters I and J to denote the sets of row and column indices, respectively, i.e.

$$I = \{1,\ldots,N\} , \quad J = \{1,\ldots,M\} .$$

The letter σ will always denote a real number from the interval $[0,1]$.

Next let $\mathcal{P}$ be the set of all mappings from J to I , and let $\mathcal{Q}$ be the set of all mappings from I to J . A sequence

$$P = \{p_\nu : p_\nu \in \mathcal{P}, \nu=1,2,\ldots\}$$

or a sequence

$$Q = \{q_\nu : q_\nu \in \mathcal{Q}, \nu=1,2,\ldots\}$$

will be referred to as response strategy of player 1 or 2 respectively. The sets of all response strategies will be denoted $\mathcal{P}^\infty$ and $\mathcal{Q}^\infty$. A response strategy P or Q such that $p_1 = p_2 = \ldots = p$ or such that $q_1 = q_2 = \ldots = q$ will be called stationary.

Let $P = \{p_\nu\} \in \mathcal{P}^\infty$ and $Q = \{q_\nu\} \in \mathcal{Q}^\infty$ be a pair of response strategies, and let n be a positive integer. With any such pair (P , Q) we now associate a quantity $H_n(P,Q|i_o)$, called the n-stage payoff given the row-predecessor $i_o \in I$, and defined by

$$H_n(P,Q|i_o) = \frac{1}{n} \sum_{\nu=1}^{n} \sigma a(i_{\nu-1},j_\nu) + (1-\sigma)a(i_\nu,j_\nu) ,$$

where $j_\nu = q_\nu(i_{\nu-1})$, $i_\nu = p_\nu(j_\nu)$, $\nu = 1,2,\ldots$.

Similarly, the n-stage payoff given the column-predecessor $j_o \in J$ is defined by

$$H_n(P,Q|j_o) = \frac{1}{n} \sum_{\nu=1}^{n} (1-\sigma)a(i_\nu,j_{\nu-1}) + \sigma a(i_\nu, j_\nu) ,$$

where this time

$$i_\nu = p_\nu(j_{\nu-1}) \ , \quad j_\nu = q_\nu(i_\nu) \ , \quad \nu = 1,2,\ldots \ .$$

The triplet $G(\sigma) = (\mathcal{P}^\infty, \mathcal{Q}^\infty, A)$ with the sequences of payoffs H_n defined as above will be referred to as matrix-differential game. (MDG)

We are now ready to state the basic definition.

Definition: If there exist a pair of response strategies $P^* \in \mathcal{P}^\infty$, $Q^* \in \mathcal{Q}^\infty$ and a real number $\Omega(\sigma)$ such that for any $i_o \in I$ or $j_o \in J$

$$\lim_{n\to\infty} H_n(P^*,Q^*|i_o) = \lim_{n\to\infty} H_n(P^*,Q^*|j_o) = \Omega(\sigma) \ , \qquad (2.1)$$

and for any $P \in \mathcal{P}^\infty$, $Q \in \mathcal{Q}^\infty$

$$\lim_{n\to\infty} \sup H_n(P,Q^*|i_o) \le \Omega(\sigma) \ , \qquad (2.2)$$

$$\lim_{n\to\infty} \sup H_n(P,Q^*|j_o) \le \Omega(\sigma) \ , \qquad (2.3)$$

$$\lim_{n\to\infty} \inf H_n(P^*,Q|i_o) \ge \Omega(\sigma) \ , \qquad (2.4)$$

$$\lim_{n\to\infty} \inf H_n(P^*,Q|j_o) \ge \Omega(\sigma) \ , \qquad (2.5)$$

then $\Omega(\sigma)$ is called the omega-value and P^* and Q^* optimal response strategies of the MDG $G(\sigma)$.

Lemma: Let $\omega, x_1,\ldots, x_n, y_1,\ldots,y_m$ be a solution of the system of $N + M$ equations:

$$x_i + \omega\sigma = \min_{j\in J}[\sigma a(i,j) + y_j] \ , \quad i \in I \ , \qquad (2.6a)$$

$$y_j + \omega(1-\sigma) = \max_{i\in I}[(1-\sigma)a(i,j)+x_i] \ , \quad j \in J \ , \qquad (2.6b)$$

let $p^* \in \mathcal{P}$ and $q^* \in \mathcal{Q}$ be such that

$$x_i + \omega\sigma = \sigma a(i,q^*(i)) + y_{q^*(i)} \ , \quad i \in I \ , \qquad (2.7a)$$

$$y_j + \omega(1-\sigma) = (1-\sigma)a(p^*(j),j) + x_{p^*(j)} \ , \quad j \in J. \qquad (2.7b)$$

Then (2.1) through (2.5) hold with $\Omega(\sigma) = \omega$ and stationary

$P^* = \{p^*,p^*,\ldots\}$, $Q^* = \{q^*,q^*,\ldots\}$.

Proof: We begin with the inequality (2.2) . By (2.6b) we have for every $k \in I$, $j \in J$

$$y_j + \omega(1-\sigma) \geq (1-\sigma)a(k,j) + x_k ,$$

so that in particular for $j = q^*(i)$

$$y_{q^*(i)} + \omega(1-\sigma) \geq (1-\sigma)a(k,q^*(i)) + x_k .$$

Substituting for $y_{q^*(i)}$ into (2.7a) this becomes

$$x_i + \omega\sigma \geq \sigma a(i,q^*(i)) + (1-\sigma)a(k,q^*(i)) + x_k - \omega(1-\sigma)$$

or

$$\omega \geq \sigma a(i,q^*(i)) + (1-\sigma)a(k,q^*(i)) + x_k - x_i , \tag{2.8}$$

for every $i \in I$, $k \in I$. Let $P = \{p_\nu\} \in \mathcal{P}^\infty$,

$i_o \in I$ be arbitrary, let

$$j_\nu = q^*(i_{\nu-1}) , \quad i_\nu = p_\nu(j_\nu) , \quad \nu = 1,2,\ldots$$

Substituting $i = i_{\nu-1}$, $k = i_\nu$ into (2.8) and averaging over $\nu = 1,\ldots,n$ we obtain

$$\omega \geq H_n(P,Q^*|i_o) + \frac{1}{n}\sum_{\nu=1}^{n} x_{i_\nu} - x_{i_{\nu-1}} .$$

But $\left|\frac{1}{n}\sum_{\nu=1}^{n} x_{i_\nu} - x_{i_{\nu-1}}\right| = \frac{1}{n}\left| x_{i_n} - x_{i_o} \right| \leq \frac{2}{n}\max_{i \in I}\{|x_i|\}$

which tends to zero as $n \to \infty$.

Hence (2.2) holds with $\Omega(\sigma) = \omega$ and $Q^* = \{q^*,q^*,\ldots\}$.

To prove (2.3) let this time $j_o \in J$ arbitrary ,

$$i_\nu = p_\nu(j_{\nu-1}) , \quad j_\nu = q^*(i_\nu) , \quad \nu = 1,2,\ldots$$

and substitute $k = i_\nu$, $i = i_{\nu+1}$ into (2.8) .

Averaging over $\nu = 1,\ldots n$ we obtain

$$\omega \geq H_n(P,Q^*|j_o) - \frac{1}{n}[(1-\sigma)a(i_1,j_o) + \sigma a(i_1,j_1)] + \frac{1}{n}\sum_{\nu=1}^{n} x_{i_\nu} - x_{i_{\nu+1}} ,$$

the last two terms again tending to zero as $n \to \infty$.

Inequalities (2.4) and (2.5) follow from (2.6a) and (2.7b) in analogous manner.

Finally to establish (2.1) we have from (2.7a,b)

$$\omega = \sigma a(i,q^*(i)) + (1-\sigma)a(p^*(q^*(i)),q^*(i)) + x_{p^*(q^*(i))} - x_i , \quad i \in I ,$$

from which by setting $i = i_\nu = p^*(j_\nu)$, $j_\nu = q^*(i_{\nu-1})$, $\nu = 1,2,...$

we have

$$\omega = H_n(P^*,Q^*|i_o) + \frac{1}{n}(x_{i_n} - x_{i_o}) ,$$

and (2.1) follows. The lemma is proved.

Remark 1: Investigation of the preceeding proof reveals that a slightly stronger statem can be made. In fact we proved that there exists a constant $C < \infty$ such that for any $P \in \mathcal{P}^\infty$, $Q \in \mathcal{Q}^\infty$

$$H_n(P,Q^*|i_o) < \Omega(\sigma) + \frac{C}{n} , \tag{2.2'}$$

$$H_n(P^*,Q|i_o) > \Omega(\sigma) - \frac{C}{n} , \tag{2.4'}$$

for every $n = 1,2,...,$ and similarly for a column - predecessor j_o. This also implies that the convergence in (2.1) is of order $O(1/n)$.

With the aid of the lemma we now prove the main theorem about MD games.

Theorem 1: Every MD game $G(\sigma)$, $\sigma \in [0,1]$ has an omega-value and each player has a stationary optimal response strategy. Further, for every $\sigma \in [0,1]$ the omega-value $\Omega(\sigma)$ is the unique number ω satisfying the system (2.6) and $P^* = \{p^*\}$, $Q^* = \{q^*\}$, where p^* and q^* satisfy (2.7a) and (2.7b) respectively, are stationary optimal response strategies.

Proof: In view of the previous lemma and the fact that, according to its definition, the omega-value must be unique, the theorem will follow as long as we prove that the system (2.6) has always a solution.

To this end notice first that if $x_1, \ldots, x_n$ is a solution of the system of N equations:

$$x_k = \min_{j\in J} \max_{i\in I} [\sigma a(k,j) + (1-\sigma)a(i,j) + x_i] - \frac{1}{N}\sum_{\ell=1}^{N} \min_{j\in J} \max_{i\in I}[\sigma a(\ell,j) + (1-\sigma)a(i,j) + x_i] , \quad k\in I , \tag{2.9}$$

then ω, $x_1,\ldots, x_N, y_1,\ldots, y_M$, where

$$\omega = \frac{1}{N}\sum_{\ell=1}^{N} \min_{j\in J} \max_{i\in I}[\sigma a(\ell,j) + (1-\sigma)a(i,j) + x_i] ,$$

$$y_j = \max_{i\in I} [(1-\sigma)a(i,j) + x_i] - \omega(1-\sigma) , \quad j\in J ,$$

is a solution of the system (2.6) .

Call temporarily $f_k(x_1,\ldots,x_N)$, $k\in I$ the right-hand side of the k-th equation (2.9) . Since

$$|f_k(x_1,\ldots,x_N)| \le \frac{1}{N}\sum_{\ell=1}^{N} \Big| \min_{j\in J} \max_{i\in I} [\sigma a(k,j) + (1-\sigma)a(i,j) + x_i] - \min_{j\in J} \max_{i\in I} [\sigma a(\ell,j) + (1-\sigma)a(i,j) + x_i]\Big|$$

$$\le \frac{1}{N}\sum_{\ell=1}^{N} \max_{j\in J}\Big|\max_{i\in I} [\sigma a(k,j) + (1-\sigma)a(i,j) + x_i] - \max_{i\in I} [\sigma a(\ell,j) + (1-\sigma)a(i,j) + x_i]\Big|$$

$$\le \frac{1}{N}\sum_{\ell=1}^{N} \max_{j\in J} \max_{i\in I}\big|[\sigma a(k,j) + (1-\sigma)a(i,j)] - [\sigma a(\ell,j) + (1-\sigma)a(i,j)]\big|$$

$$\le \frac{\sigma}{N}\sum_{\ell=1}^{N} \max_{j\in J}|a(k,j) - a(\ell,j)| \le 2\sigma \, \| A \| ,$$ where

$$\| A \| = \max_{i \in I} \max_{j \in J} |a(i,j)| ,$$

and since clearly $\sum_{k=1}^{N} f_k(x_1,\ldots,x_N) = 0$,

the vector-valued function $\underline{f} = (f_1,\ldots,f_N)$ maps the (N-1)-dimensional hypercube

$$C = \{(x_1,\ldots,x_N) : \sum_{i=1}^{N} x_i = 0 , \quad \max_{i \in I} |x_i| \leq 2\sigma \| A \| \}$$

into itself. Next for any $k \in I$

$$|f_k(x_1,\ldots,x_N) - f_k(x_1',\ldots,x_N')| \leq \Big| \min_{j \in J} \max_{i \in I}$$

$$[\sigma a(i,j) + (1-\sigma)a(k,j) + x_i] - \min_{j \in J} \max_{i \in I} [\sigma a(i,j)$$

$$+ (1-\sigma)a(k,j) + x_i'] \Big| + \frac{1}{N} \sum_{\ell=1}^{N} \Big| \min_{j \in J} \max_{i \in I}$$

$$[\sigma a(\ell,j) + (1-\sigma)a(i,j) + x_i'] - \min_{j \in J} \max_{i \in I} [\sigma a(\ell,j) + (1-\sigma)a(i,j) + x_i] \Big|$$

$$\leq \max_{i \in I} |x_i - x_i'| + \frac{1}{N} \sum_{\ell=1}^{N} \max_{i \in I} |x_i' - x_i| = 2 \max_{i \in I} |x_i - x_i'| .$$

Hence the function $\underline{f}$ is also continuous and therefore, by Brouwer Fixed Point Theorem there exists $(x_1,\ldots,x_N) \in C$ such that

$$x_k = f_k(x_1,\ldots,x_N) , \quad k \in I .$$

Thus the system (2.9) and consequently the system (2.6) has always a solution and the theorem is proved.

John Danskin defined originally the omega-value of a MD game as a limit of ordinary pure values of a sequence of games with perfect information ([2], see also [3]). The next theorem shows that Danskin's definition and the one used in this paper are equivalent.

Theorem 2: If $\Omega(\sigma)$ is the omega-value of a MD-game $G(\sigma)$ then for every $i_o \in I$

$$\Omega(\sigma) = \lim_{n\to\infty} \min_{j_1\in J} \max_{i_1\in I} \dots \min_{j_n\in J} \max_{i_n\in I} \frac{1}{n} \sum_{\nu=1}^{n} \sigma a(i_{\nu-1},j_\nu) + (1-\sigma)a(i_\nu,j_\nu) , \tag{2.10}$$

and for every $j_o \in J$

$$\Omega(\sigma) = \lim_{n\to\infty} \max_{i_1\in I} \min_{j_1\in J} \dots \max_{i_n\in I} \min_{j_n\in J} \frac{1}{n} \sum_{\nu=1}^{n} (1-\sigma)a(i_\nu,j_{\nu-1}) + \sigma a(i_\nu,j_\nu) . \tag{2.11}$$

Proof: Denote temporarily

$$W_n(i_o,j_1,i_1,\dots,j_n,i_n) = \frac{1}{n} \sum_{\nu=1}^{n} \sigma a(i_{\nu-1},j_\nu) + (1-\sigma)a(i_\nu,j_\nu).$$

Let $p^* \in \mathcal{P}$ and $q^* \in \mathcal{Q}$ be as in (2.7) . By (2.2') and (2.4') of Remark 1 we have for every $n = 1,2,\dots$

$$W_n(i_o,q^*(i_o),i_1,q^*(i_1),\dots,q^*(i_{n-1}),i_n) < \Omega(\sigma) + \frac{C}{n} \tag{2.12}$$

for any sequence of row indices $i_o,i_1,\dots$, and

$$W_n(i_o,j_1,p^*(j_1),\dots,j_n,p^*(j_n)) > \Omega(\sigma) - C/n , \tag{2.13}$$

for any sequence of column indices $j_1,j_2,\dots$ and any $i_o \in I$.

From (2.12) we obtain successively

$$\max_{i_n} W_n(i_o,q^*(i_o),\dots,q^*(i_{n-1})i_n) < \Omega(\sigma) + \frac{C}{n}$$

$$\min_{j_n} \max_{i_n} W_n(i_o,q^*(i_o),\dots,i_{n-1},j_n,i_n) < \Omega(\sigma) + \frac{C}{n}$$

$$\dots$$

$$\min_{j_1} \max_{i_1} \dots \min_{j_n} \max_{i_n} W_n(i_o,j_1,\dots,j_n,i_n) < \Omega(\sigma) + \frac{C}{n} .$$

Similarly from (2.13) we obtain eventually

$$\min_{j_1} \max_{i_1} \dots \min_{j_n} \max_{i_n} W_n(i_o,j_1,\dots,j_n,i_n) > \Omega(\sigma) + \frac{C}{n}$$

and (2.10) follows. (2.11) is proved in the same fashion.

To the end of this section let us investigate some simple properties of the omega-value. From the definition or from the previous theorem it is immediately obvious that

$$\Omega(0) = \min_{j\in J} \max_{i\in I} a(i,j) ,$$

$$\Omega(1) = \max_{i\in I} \min_{j\in J} a(i,j) .$$

Also, it is easy to see that if the matrix A is reduced by successively eliminating dominated rows and columns the omega-value $\Omega(\sigma)$ is not affected. Another obvious property is that if the entries $a(i,j)$ are all multiplied by a positive constand α and/or an arbitrary constant β is added to all of them the omega-value changes accordingly while the optimal response strategies remain unchanged. Some less obvious properties of the omega-value are given in the following theorem.

Theorem 3: The omega-value $\Omega(\sigma)$ of a MD game is a continuous, non-increasing and finite piece-wise linear function of $\sigma \in [0,1]$.

Proof: Let $\sigma_1 \in [0,1]$, $\sigma_2 \in [0,1]$. By Theorem 2

$$|\Omega(\sigma_1) - \Omega(\sigma_2)| = \Big|\lim_{n\to\infty}\Big[\min_{j_1\in J} \max_{i_1\in I} \dots \min_{j_n\in J} \max_{i_n\in I}$$

$$\frac{1}{n}\sum_{\nu=1}^{n} \sigma_1 a(i_{\nu-1},j_\nu) + (1-\sigma_1)a(i_\nu,j_\nu) - \min_{j_1\in J} \max_{i_1\in I}$$

$$\dots \min_{j_n\in J} \max_{i_n\in I} \frac{1}{n}\sum_{\nu=1}^{n} \sigma_2 a(i_{\nu-1},j_\nu) + (1-\sigma_2)a(i_\nu,j_\nu)\Big]\Big|$$

$$\leq \lim_{n\to\infty} \Big| \min_{j_1\in J} \max_{i_1\in I} \cdots \min_{j_n\in J} \max_{i_n\in I} \frac{1}{n} \sum_{\nu=1}^{n} \sigma_1 a(i_{\nu-1},j_\nu)$$

$$+(1-\sigma_1)a(i_\nu,j_\nu) - \min_{j_1\in J} \max_{i_1\in I} \cdots \min_{j_n\in J} \max_{i_n\in I}$$

$$\frac{1}{n} \sum_{\nu=1}^{n} \sigma_2 a(i_{\nu-1},j_\nu) + (1-\sigma_2)a(i_\nu, j_\nu) \Big|$$

$$\leq \lim_{n\to\infty} \max_{j_1\in J} \max_{i_1\in I} \cdots \max_{j_n\in J} \max_{i_n\in I} \frac{1}{n} \sum_{\nu=1}^{n} \Big| (\sigma_1-\sigma_2)$$

$$a(i_{\nu-1},j_\nu) + (\sigma_2-\sigma_1)a(i_\nu,j_\nu) \Big|$$

$$\leq 2 \, \| A \| \, |\sigma_1-\sigma_2| \, , \quad \text{where again} \quad \| A \| = \max_{i\in I} \max_{j\in J} | a(i,j) | \, .$$

Hence $\Omega(\sigma)$ is continuous in $\sigma\in[0,1]$.

Next let

$$(i_1 j_1, \ldots, i_m, j_m)$$

be a finite sequence of distinct row and column indices such that

$$j_\nu = q^*(i_\nu) \, , \quad i_{\nu+1} = p^*(j_\nu) \, , \quad \nu = 1,\ldots,m \, , \quad i_{m+1} = i_1 \, ,$$

where $p^*\in\mathcal{P}$ and $q^*\in\mathcal{Q}$ satisfy (2.7) . Clearly, such a sequence exists for every $\sigma\in[0,1]$ and may be called an optimal cycle. By (2.7) we have

$$x_{i_\nu} + \omega\sigma = \sigma a(i_\nu,j_\nu) + y_{j_\nu} \quad , \tag{2.14a}$$

$$y_{j_\nu} + \omega(1-\sigma) = (1-\sigma)a(i_{\nu+1},j_\nu) + x_{i_{\nu+1}} \quad , \quad \nu = 1,\ldots,m \tag{2.14b}$$

Adding these equations and using the fact that $x_{i_{m+1}} = x_1$ we obtain

$$m\omega = \sigma \sum_{\nu=1}^{m} a(i_\nu, j_\nu) + (1-\sigma) \sum_{\nu=1}^{m} a(i_{\nu+1}, j_\nu) \ . \tag{2.15}$$

Since an optimal cycle exists for every $\sigma \epsilon [0,1]$ and since there is only a finite number of possible optimal cycles continuity of $\Omega(\sigma)$ implies that there must be a finite partition

$$0 = \sigma_o < \sigma_1 < \ldots < \sigma_p = 1$$

of the interval $[\sigma,1]$ such that in each interval $[\sigma_{k-1}, \sigma_k]$ the omega-value is a linear function of σ, in particular

$$\Omega(\sigma) = \frac{\sigma}{m} \sum_{\nu=1}^{m} a(i_\nu, j_\nu) + \frac{1-\sigma}{m} \sum_{\nu=1}^{m} a(i_{\nu+1}, j_\nu) \ , \quad \sigma_{k-1} \leq \sigma \leq \sigma_k \ . \tag{2.16}$$

Finally, since by (2.6b)

$$y_j + \omega(1-\sigma) \geq (1-\sigma)a(k,j) + x_k$$

for every $k \epsilon I$, $j \epsilon J$ we have by setting $j = j_\nu$, $k = i_\nu$

$$y_{j_\nu} + \omega(1-\sigma) \geq (1-\sigma)a(i_\nu, j_\nu) + x_{i_\nu} \ , \ \nu = 1,\ldots,m \ ,$$

from which by adding these inequalities to the m equations (2.14a) we obtain

$$m\omega \geq \sigma \sum_{\nu=1}^{m} a(i_\nu, j_\nu) + (1-\sigma) \sum_{\nu=1}^{m} a(i_\nu, j_\nu) \ . \tag{2.17}$$

Comparing (2.17) with (2.15) we see that

$$\sum_{\nu=1}^{m} a(i_\nu, j_\nu) \leq \sum_{\nu=1}^{m} a(i_{\nu+1}, j_\nu) ,$$

and hence by (2.16) $\Omega(\sigma_{k-1}) \geq \Omega(\sigma_k)$. Thus the function $\Omega(\sigma)$ is non-increasing and the theorem is proved.

3. Ergodic Pairs and a Method for Solving MDG's

Given any starting point and any pair of stationary response strategies, the (row,column) pair will eventually repeat itself, since there are only finitely many such pairs, and a minimal cycle of such pairs will then repeat over and over. The payoff to player I will be the average payoff over that cycle (2.16). If the cycle turns out not to depend on the initial row or column, the pair of stationary response strategies will be termed ergodic, and such a pair will be referred to as an ESP. An optimal pair of stationary response strategies will be an OSP for whatever values of σ the pair is optimal. Note that whether or not a pair of strategies is an ESP has nothing to do with the payoff matrix or σ.

Since the Ω-value does not depend on the starting point, an OSP must be an ESP, except when an OSP has multiple cycles each of which has an average payoff (formula 2.16) of $\Omega(\sigma)$. Such OSP's will turn out to be of some importance to our method, but let us ignore them for the moment, and attempt to solve for the Ω-value (we want the entire function $\Omega(\cdot)$, not $\Omega(\sigma)$ for some particular σ) using only ESP's.

Our method will be to first solve for $\Omega(0)$, and then attempt to find $\Omega(\sigma)$ in adjoining intervals until finally we find an interval that includes $\sigma = 1$. When $\sigma = 0$, $\Omega(0) = \min_{j\in J} \max_{i\in I} a(i,j)$, $p^*(j)$ is a row with the largest payoff in the j^{th} column, and $q^*(i) \equiv j^*$ is a min max column. (p^*,q^*) is an ESP, since the only possible cycle with (p^*,q^*) is $\{(p^*(j^*),j^*),(p^*(j^*),j^*)\}$. Furthermore, (p^*,q^*) will be an OSP in some maximal interval $[0,\sigma_1]$, where possibly $\sigma_1 = 0$. In order to find σ_1, we must solve equations (2.7) for $\bar{x}$, $\bar{y}$, and ω as functions of σ, after which σ_1 will be the largest value of σ for which (2.6) holds.

Before solving (2.7), we first note that an arbitrary function of σ can always be added to both sides of 2.6b and 2.7b. In order to obtain linear expressions, it is convenient to let this function be $\omega\sigma$. If we then let $x_i' = x_i + \omega\sigma$, 2.6 and 2.7 become

$$x_i' = \min_{j\in J}[\sigma\ a(i,j) + y_j],\ i\in I \qquad (3.6a)$$

$$y_j + \omega = \max_{i\in I}[(1-\sigma)\ a(i,j) + x_i'],\ j\in J \qquad (3.6b)$$

$$x_i = \sigma\ a(i,q^*(i)) + y_{q^*(i)} \quad i\in I \qquad (3.7a)$$

$$y_j + \omega = (1-\sigma)\ a(p^*(j),j) + x^1_{p^*(j)} \quad j\in J \qquad (3.7b)$$

In the following, we will drop the $'$ on x_i', and will not refer again to (2.6) and (2.7).

Lemma: Let (p^*,q^*) be any ESP. Then the solution of (3.7) for $\bar{x}$, $\bar{y}$, ω as functions of σ is unique for ω, and unique for $\bar{x}$, $\bar{y}$ except that one x_i or y_j can be an arbitrary function of σ. If the arbitrary function is linear, so are $\bar{x}$ and $\bar{y}$. The quantity ω is a linear function of σ in any case.

Proof: Let the unique cycle be $(i_1,j_1,\ldots,i_m,j_m)$. Let $x_{i_1} = \subset(\sigma)$, an arbitrary function. (3.7a) then defines y_{j_1}, after which (3.7b) defines x_{i_2}, etc. Using this procedure, we can define x(y) numbers for each row (column) in the cycle. Furthermore, (3.7b) with $j = j_m$ will be satisfied if ω is obtained from formula (2.16). This defines all x(y) numbers for rows (columns) that appear in the cycle in a manner consistent with (3.7). Let i_1 be a row that does not appear in the cycle, and let $(i_1,j_1,i_2,j_2,\ldots,i_m,j_m)$ be such that $j_\gamma = q^*(i_\gamma)$, $i_{v+1} = p^*(j_v)$, and j_m is the only column in the cycle (such a j_m will always exist, on account of the uniqueness of the cycle). Since y_{j_m} is defined, we can use (3.7a) to obtain x_{i_m}, then (3.7b) to obtain $y_{j_{m-1}}$, etc., until finally we obtain x_{i_1}. Furthermore, since any other such sequence that contains i_1 will also contain $j_i,\ldots,j_m$, there is no danger of obtaining a conflicting definition of x_i in the process of defining some other x_i or y_j. Similar remarks hold for columns that do not appear in the cycle. Every x_i and y_j has therefore been uniquely defined in a manner consistent with (3.7). If $\subset(\sigma)$ is linear, the linearity of $\bar{x}$ and $\bar{y}$ follows from the linearity of (3.7). Since (2.16) is also linear, the lemma is proved.

We are now ready to describe the process of interval extension, supposing that $\Omega(\sigma)$ is already known for $0 \le \sigma \le \sigma^*$. If the OSP (p^*,q^*) at $\sigma = \sigma^*$ is actually an ESP, x_i, y_j and ω will satisfy (3.6) at $\sigma = \sigma^*$. Therefore, unless there are ties for the minimum in (3.6a) or the maximum in (3.6b) when $\sigma = \sigma^*$, the same expressions will satisfy (3.6) for $\sigma^* \le \sigma \le \sigma^{**}$, where $\sigma^{**} > \sigma^*$; that is, (p^*,q^*) is actually an OSP for $\sigma^* \le \sigma \le \sigma^{**}$. The quantity σ^{**} will be the smallest σ for which there is a tie. For definiteness, suppose there is a tie in (3.6a), so that

$$x_i = \sigma^{**}a(i,j) + y_j = \sigma^{**}a(i,q^*(i)) + y_{q^*(i)},$$

where $j \neq q^*(i)$. We form a new response strategy $q^{**}(\cdot)$ by defining $q^{**}(i) = j$, and $q^{**}(k) = q^*(k)$ for $k \neq i$. Then (p^*,q^{**}) is an OSP at $\sigma = \sigma^{**}$. If it turns out that (p^*,q^{**}) is also an ESP, then we can repeat the process of solving (3.7) for $x_i(\sigma)$, $y_i(\sigma)$, and $\omega(\sigma)$, increasing σ until another tie is encountered in (3.6), etc.

Since we have a starting point (an OSP that is also an ESP at $\sigma = 0$), there is some hope that the process might map out $\Omega(\sigma)$ for $0 \leq \sigma \leq 1$ in intervals, with an OSP/ESP for each interval. Two things are clear:

1) The process is failsafe in the sense that all answers are correct.
2) The process may not provide an answer. This could happen either because the process gets stuck at a certain σ (multiple ties might cause this), or because (p^*,q^{**}) is at some stage not and ESP.

Computational experience with matrices chosen at random has shown that the process will always (?) provide the answer for (3 x 3) matrices, but that it will sometimes fail on (9 x 9) matrices and will nearly always fail on (18 x 18) matrices. When it fails, it fails because it discovers an OSP that is not an ESP.

This reason for failure is somewhat suprising. If an OSP that is not an ESP is to hold over an <u>interval</u>, then (2.16) must be the same function of σ for two or more disjoint cycles. If the a(i,j) are chosen at random, the probability of this is 0. In fact, the probability is 0 that there could be an OSP with three or more cycles for <u>any</u> value of σ. However, the probability is not zero that there can exist particular values of σ at which an OSP has exactly two disjoint cycles, and it is this type of OSP that the above procedure tends to discover. The problem, then, is this: Given ESP_1 that is an OSP over some interval $[\sigma^*,\sigma^{**}]$ and OSP_1 at σ^{**} that is not an ESP, how can we discover an ESP_2 that is OSP over $[\sigma^{**},\sigma^{***}]$ where $\sigma^{***} > \sigma^{**}$?

From ESP_1, we can obtain numbers x_i, y_j, and ω that satisfy (3.6) at $\sigma = \sigma^{**}$. There is a tie in one of the equations (3.6), which we assume for convenience to be in (3.6a):

$$x_i = \sigma^{**}a(i,j_1) + y_{j_1} = \sigma^{**}a(i,j_2) + y_{j_2},$$

where

$$j_1 = q^*(i) \quad \text{and} \quad j_2 = q^{**}(i),$$

$$ESP_1 = (p^*,q^*), \quad \text{and} \quad OSP_1 = (p^*,q^{**}).$$

We assume OSP_1 has exactly two cycles, one of which must be the ESP_1 cycle. The rows and columns can be partitioned into S_1 and S_2, where S_1 includes all rows and columns in the ESP_1 cycle, plus all those rows and columns that OSP_1 maps into the ESP_1 cycle, and S_2 is defined similarly for the other cycle. Define $x(\delta)$ and $y(\delta)$ by

$$x_i(\delta) = \begin{cases} x_i & \text{if } i \in S_1 \\ x_i - \delta & \text{if } i \in S_2 \end{cases}$$

$$y_j(\delta) = \begin{cases} y_j & \text{if } j \in S_1 \\ y_j - \delta & \text{if } j \in S_2 \end{cases}$$

Since (3.7) with OSP_1 never compares a row in S_1 with a column in S_2, or a row in S_2 with a column in S_1, $\bar{x}(\delta),\bar{y}(\delta)$ and OSP_1 will solve (3.7) regardless of δ. Let $\hat{\delta}$ be the largest δ such that $\bar{x}(\delta),\bar{y}(\delta)$ satisfy (3.6). Since we have assumed there is only one tie when $\delta = 0$, $\hat{\delta} > 0$. For $0 < \delta < \hat{\delta}$, there are no ties. When $\delta = \hat{\delta}$, there is at least one tie, and we assume there is exactly one. Resolving the tie one way, we obtain OSP_1. Resolving the tie the other way, we obtain OSP_2 different from OSP_1 and ESP_1. If OSP_2 is actually ESP_2, we re-solve (3.7) and proceed with interval extension. If OSP_2 is not ESP, then with probability one it has the same two cycles as OSP_1, since otherwise there would be three distinct cycles with the same average payoff in the sense (2.16). Let S_1^1 be all those rows and columns that OSP_2 maps to the first (original ESP_1) cycle, and similarly for S_2^1. Now repeat the process of subtracting δ from every x_i or y_j with subscript in S_2^1 until still a different tie is revealed, with corresponding OSP_3, etc. Sooner or later a new ESP will be discovered, and the basic process of interval extension can continue.

The above is not intended to be a proof that the procedure will work, but only as an explanation of the process used by a FORTRAN program called MATDIF to solve differential games. MATDIF is failsafe, in the sense that it deals only with solutions of (3.6) and (3.7), but it has not been proved that MATDIF will always provide an answer for all $\sigma \in [0,1]$. However, MATDIF has not failed to produce an answer for any of the approximately 100 test matrices with elements chosen to be uniform random numbers between 0 and 100. The program compiles in about 10 secs on the NPS IBM360/67, and the run time is approximately $120(M/50)^{1.5}(N/50)^{1.5}$

seconds (see Table 1). The program is available from Washburn. In the future, a proof that MATDIF or a procedure modified to account for degeneracies will always provide an answer will be provided.

$$T(M,N) \approx 120\left(\frac{M}{50}\right)^{1.5}\left(\frac{N}{50}\right)^{1.5}$$

M \ N	9	18	50
9	.6 seconds (20 runs)	1.5 seconds (30 runs)	
18	1.5 seconds (30 runs)	4.5 seconds (2 runs)	
50	10 seconds (10 runs)		120 seconds (3 runs)

Table 1 Average run times for MATDIF in seconds

Example: Figure 1 shows $\Omega(\sigma)$ for a typical 18 x 18 game. There are 20 distinct linear segments in this case. MATDIF actually considers 70 different ESP's, but many of the (distinct) ESP's have the same cycle. The solution proceeds by interval extension up to $\sigma = \sigma_1 \equiv .53169$. The ESP_1 that is OSP at σ_1 and for slightly smaller values is illustrated in figure 2; each column has an "x" corresponding to I's choice in the column, and each row has an "o" corresponding to II's choice in the row. So x's move horizontally to o's, and o's move vertically to x's. The only cycle is shown solid in the figure.

The OSP_1 following ESP_1 is identical to ESP_1 except that the 0 in row 16 is moved from column 11 to column 2. A new cycle forms, shown as a dashed line. Both cycles have the same average payoff. The reader might amuse himself by delineating S_1 and S_2. OSP_2 is formed by moving the 0 in row 6 from column 3 to column 6, and still has the same two cycles. OSP_3 is formed by moving the x in column 16 from row 16 to row 10, and also has the same two cycles. Finally, OSP_4 is formed by moving the 0 in row 5 from column 17 to column 8, leaving only the dashed cycle and hence a new ESP. This ESP is valid for $\sigma_1 \leq \sigma \leq .5402$, and only ESP's are encountered for larger σ. The run time for solving this game is 4.12 seconds.

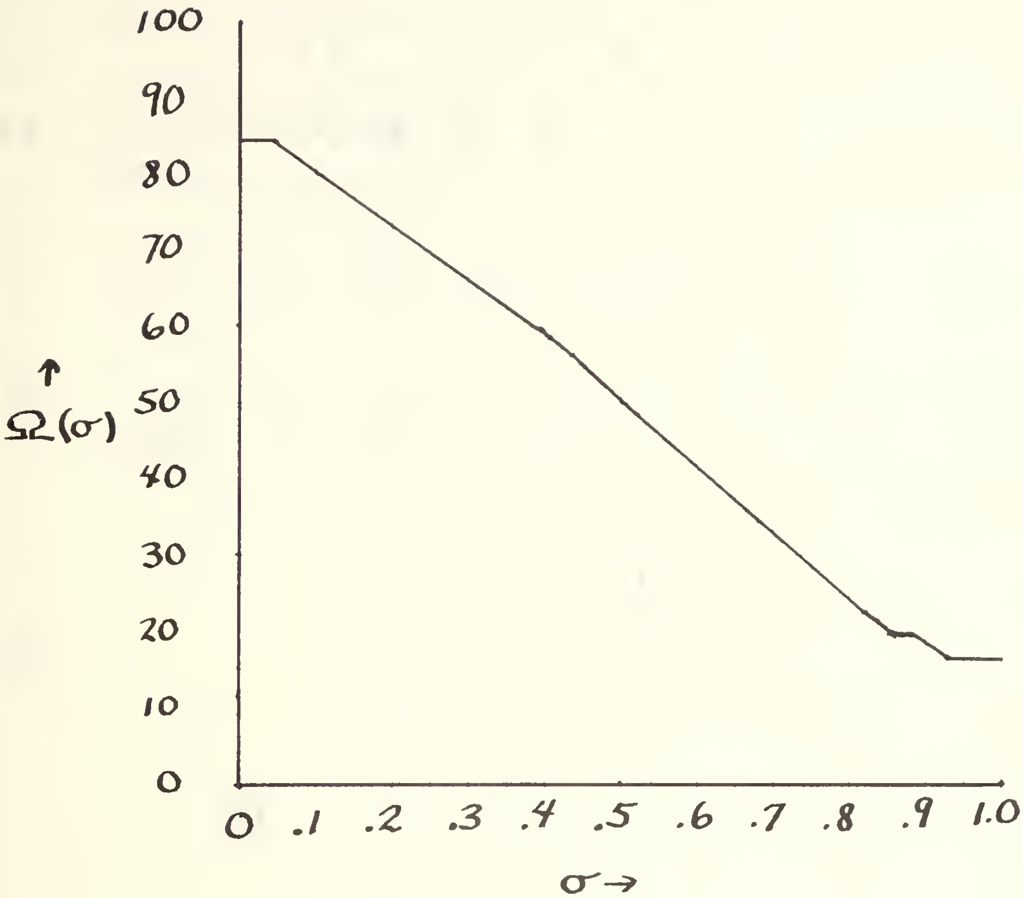

FIGURE 1

$\Omega(\sigma)$ for an 18×18 game

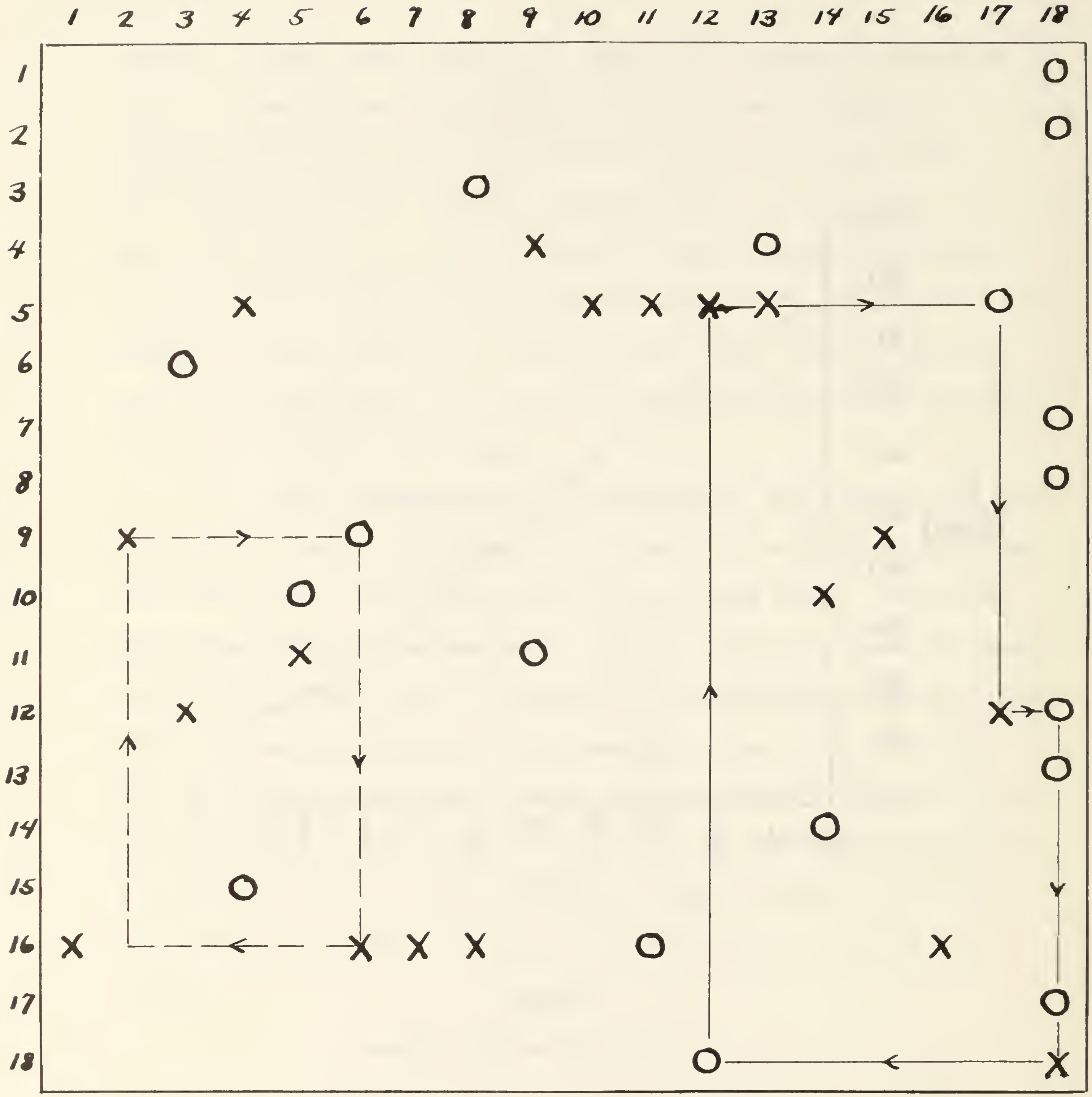

FIGURE 2

Illustration of cycles at σ = .53169
with same game as in FIGURE 1.

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